and is 0.028% of the center deflection. Maximum deviation from zero normal slope occurs for the square and is 0.12% of the maximum slope within the plate. Comparing with results obtained when the problems are solved using only five equally spaced boundary points, we find maximum changes in $w(0, \theta)$, $M(0, \theta)$, and $M_x(a, 0)$ to be 0.006, 0.004, and 0.2%, respectively.

Significant results for the case when all sides are simply supported are presented in Table 2. The maximum deviation from zero deflection along the boundary is for the heptagon and is 0.15% of the center deflection. Maximum deviation from zero normal moment along the boundary occurs in the case of the hexagon and is 16.1% of the moment at the center. Although the moment deviation appears to be large, the residual moments along the edges oscillate to be essentially self-equilibrating and, by St. Venant's principle, should affect the moment at the center to a lesser extent. There is also more change in the results between solutions using five and nine boundary points for the simply supported case. The maximum changes in $w(0, \theta)$, $M(0, \theta)$, and $\partial w/\partial x(a, 0)$ are 2.4, 2.3, and 3.1%, respectively. For these reasons, the figures presented in Table 2 are given to fewer significant figures than those in Table 1.

To the authors' knowledge no other published results are available for comparison except for the triangular and square shapes. A partial summary of deflections and moments for these two cases is readily available in Ref. 2. The values in Tables 1 and 2 agree closely with those of Ref. 2 except for the value of bending moment at the center of a clamped triangular plate. Reference 2 presents differing values of M_x and M_y at $r = 0$, whereas, from considerations of symmetry and Mohr's circle, the bending moment at the center must be the same in all directions for any of the regular polygonal plate problems being presently considered.

Another result having interesting geometrical significance is that a mathematical maximum for bending moment occurs at the center for all shapes except the triangle. In the clamped case the mathematical maximum within the plate is $M_y = 0.1028\ q_0a^2$, occurring at $x = -0.210\ a$, $y = 0$ (see Fig. 1); in the simply supported case the maximum is $M_y = 0.233\ q_0a^2$, occurring at $x = -0.388\ a$, $y = 0$.

A final item of interest to be noted from Table 2 is that the deflection of simply supported, uniformly loaded regular polygonal plates is always less than that for their inscribing circles except for $m = 3$ and 4. This is because the conditions of zero tangential slope around the boundary of a polygonal plate result in it being clamped at its corners. Indeed the deflection steadily decreases as m increases from 3 to 10. Beyond $m = 10$, the effect of adding additional corners to the plate becomes overshadowed by the reduction in "clamping effectiveness" per corner because of the large interior angles present, and ultimately, as $m \to \infty$, the solution converges to that for the simply supported circle rather than the clamped one.

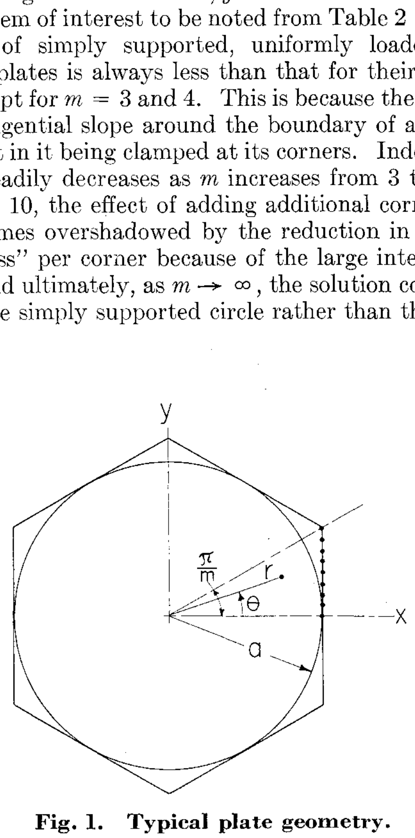

Fig. 1. Typical plate geometry.

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Distributed Mass Matrix for Plate Element Bending

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A METHOD of deriving distributed mass matrices for structural components has been given in Ref. 1. The method has been applied to the bending of beam elements in Ref. 1 and has been extended to include transverse shear and rotary inertia effects in Ref. 2.

The purpose of the present note is to provide the ingredients for forming a distributed mass matrix for plate element bending. Use of such a matrix in the natural mode analysis of plate and shell structures should significantly improve the accuracy of the results obtained using a diagonal mass matrix. Such improvements have already been demonstrated for beams.[1]

The kinetic energy of a plate element may be written

$$T = \frac{1}{2}\int_A m(x,y)\dot{d}^2 dA$$

where d represents the total displacement of the elemental area dA. In the Rayleigh-Ritz technique, the deformation of the system is represented by a set of unit displacement functions φ_i with amplitudes q_i. The expression for d is written

$$d(x,y,t) = \sum_i \varphi_i(x,y)\ q_i(t)$$

The q_i are the generalized displacements. For the plate there are 12 of these; the vertical deflection w and two coordinate slopes w_x and w_y at each of the four corners.

Substituting into T gives

$$T = \frac{1}{2}\sum_{i,j}\int_A m\varphi_i\varphi_j\dot{q}_i\dot{q}_j\ dA = \frac{1}{2}\sum_{i,j} m_{ij}\dot{q}_i\dot{q}_j$$

so that the elements of the mass matrix are determined from

$$m_{ij} = \int_A m(x,y)\varphi_i(x,y)\varphi_j(x,y)\ \ dA$$

To determine the φ_i, assume a form for the translation w. This may be written

$$w = \sum_i c_i\theta_i = c'\theta = \theta' c$$

The c_i are undetermined coefficients, and the θ_i are suitable polynomials. Although many choices for these are possible, only certain ones can be manipulated and also lead to satisfactory results. One such set is[3]: $\theta_1 = 1$, $\theta_2 = x$, $\theta_3 = y$, $\theta_4 = x^2/2$, $\theta_5 = xy$, $\theta_6 = y^2/2$, $\theta_7 = x^3/6$, $\theta_8 = x^2y/2$, $\theta_9 = xy^2/2$, $\theta_{10} = y^3/6$, $\theta_{11} = x^3y/6$, $\theta_{12} = xy^3/6$. With this expression for w, the generalized displacements may be written in the matrix form $q = \pi c$, where the elements of π are the coordinates of the four corners (Fig. 1) of the plate. Substituting back into the form for w after inversion of π gives $w = \theta'\pi^{-1}q$. The unit displacement functions φ_i are then determined as the translation w with $q_i = 1$ and all other generalized displacements equal to zero.

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The π^{-1} matrix is

$$
\pi^{-1} = \begin{array}{cc}
\begin{array}{cccccccccccc} w_1 & w_{x_1} & w_{y_1} & w_2 & w_{x_2} & w_{y_2} & w_3 & w_{x_3} & w_{y_3} & w_4 & w_{x_4} & w_{y_4} \end{array} & \\
\begin{bmatrix}
\frac{1}{4} & -\frac{a}{8} & \frac{b}{8} & \frac{1}{4} & -\frac{a}{8} & -\frac{b}{8} & \frac{1}{4} & \frac{a}{8} & -\frac{b}{8} & \frac{1}{4} & \frac{a}{8} & \frac{b}{8} \\
\frac{3}{8\,a} & -\frac{1}{8} & \frac{b}{8\,a} & \frac{3}{8\,a} & -\frac{1}{8} & -\frac{b}{8\,a} & -\frac{3}{8\,a} & -\frac{1}{8} & \frac{b}{8\,a} & -\frac{3}{8\,a} & -\frac{1}{8} & -\frac{b}{8\,a} \\
-\frac{3}{8\,b} & \frac{a}{8\,b} & -\frac{1}{8} & \frac{3}{8\,b} & -\frac{a}{8\,b} & -\frac{1}{8} & \frac{3}{8\,b} & \frac{a}{8\,b} & -\frac{1}{8} & \frac{3}{8\,b} & -\frac{a}{8\,b} & -\frac{1}{8} \\
0 & \frac{1}{4\,a} & 0 & 0 & \frac{1}{4\,a} & 0 & 0 & -\frac{1}{4\,a} & 0 & 0 & -\frac{1}{4\,a} & 0 \\
-\frac{1}{2\,ab} & \frac{1}{8b} & -\frac{1}{8\,a} & \frac{1}{2\,ab} & -\frac{1}{8\,b} & -\frac{1}{8\,a} & -\frac{1}{2\,ab} & -\frac{1}{8\,b} & \frac{1}{8\,a} & \frac{1}{2\,ab} & \frac{1}{8\,b} & \frac{1}{8\,a} \\
0 & 0 & -\frac{1}{4\,b} & 0 & 0 & \frac{1}{4\,b} & 0 & 0 & \frac{1}{4\,b} & 0 & 0 & -\frac{1}{4\,b} \\
-\frac{3}{4\,a^3} & \frac{3}{4\,a^2} & 0 & -\frac{3}{4\,a^3} & \frac{3}{4\,a^2} & 0 & \frac{3}{4\,a^3} & \frac{3}{4\,a^2} & 0 & \frac{3}{4\,a^3} & \frac{3}{4\,a^2} & 0 \\
0 & -\frac{1}{4\,ab} & 0 & 0 & \frac{1}{4\,ab} & 0 & 0 & -\frac{1}{4\,ab} & 0 & 0 & \frac{1}{4\,ab} & 0 \\
0 & 0 & -\frac{1}{4\,ab} & 0 & 0 & \frac{1}{4\,ab} & 0 & 0 & -\frac{1}{4\,ab} & 0 & 0 & \frac{1}{4\,ab} \\
\frac{3}{4\,b^3} & 0 & \frac{3}{4\,b^2} & -\frac{3}{4\,b^3} & 0 & \frac{3}{4\,b^2} & -\frac{3}{4\,b^3} & 0 & \frac{3}{4\,b^2} & \frac{3}{4\,b^3} & 0 & \frac{3}{4\,b^2} \\
\frac{3}{4\,a^3b} & -\frac{3}{4\,a^2b} & 0 & -\frac{3}{4\,a^3b} & \frac{3}{4\,a^2b} & 0 & \frac{3}{4\,a^3b} & \frac{3}{4\,a^2b} & 0 & -\frac{3}{4\,a^3b} & -\frac{3}{4\,a^2b} & 0 \\
\frac{3}{4\,ab^3} & 0 & \frac{3}{4\,ab^2} & -\frac{3}{4\,ab^3} & 0 & \frac{3}{4\,ab^2} & \frac{3}{4\,ab^3} & 0 & -\frac{3}{4\,ab^2} & -\frac{3}{4\,ab^3} & 0 & -\frac{3}{4\,ab^2}
\end{bmatrix} &
\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \end{array}
\end{array}
$$

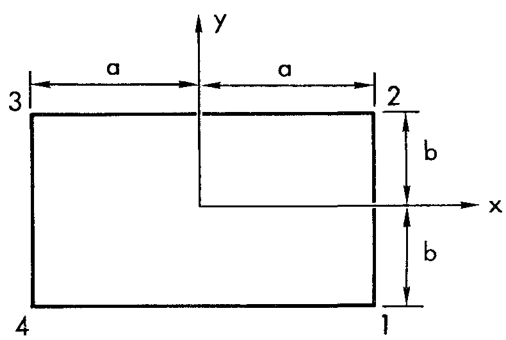

Fig. 1 Plate element.

Then, for example,

$$\varphi_1 = \tfrac{1}{4} + (3/8a)\,x - (3/8b)y - (1/2ab)xy -$$
$$(1/8a^3)x^3 + (1/8b^3)y^3 + (1/8a^3b)x^3y + (1/8ab^3)xy\text{-}$$

With the origin of the plate element at the center, the functions that are even in either x or y, after integration, will vanish upon evaluation and so need not be retained in the integrand.

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Order of a Perturbation Method

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Introduction

It is shown that accuracies usually denoted as "second order" (or higher) can be achieved by repeated application of "first-order" expressions, rather than by derivation of second-order expressions. The difference in approach is analogous to that between existence proofs for solutions of differential equations using Picard iterants instead of dominating series. A numerical example is given. It is postulated that one reason the second-order expressions derived for "Telstar" orbit prediction[2] are apparently at least ten times more accurate than other second-order methods[5] is this difference in the method of application of the expressions.

Proof

Equations of the type

$$da_i/dt = \kappa G_i(a_j,t) \qquad i,j, = 1,\ldots,6 \tag{1}$$

where κ is a small parameter, arise quite naturally in celestial

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